Christ Church Grammar School

2019
TEST 5

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	2 minutes
Working time for this section:	15 minutes
Marks available:	13 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

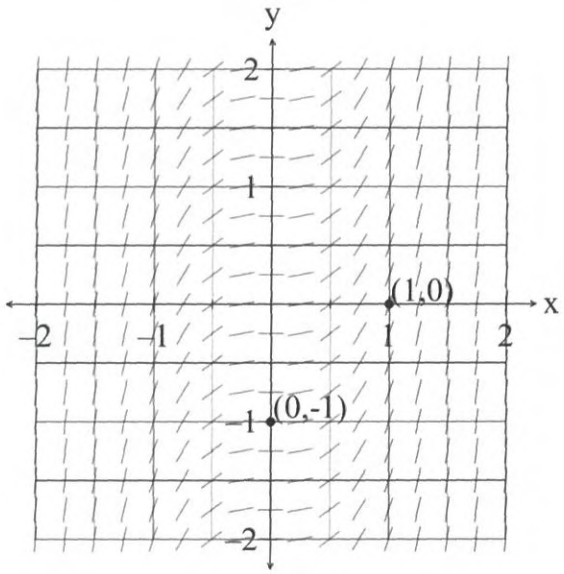
6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 1 **(5 marks)**

A first order differential equation has a slope field as shown in the diagram below



(a) Determine a general differential equation that is represented by this slope field. (2 marks)

$$\frac{dy}{dx} = ax^2$$

(b) Find the unique solution to the differential equation in part (a) given that the points $(1, 0)$ and $(0, -1)$ belong to the curve. (3 marks)

$$\int dy = \int ax^2 dx \quad \checkmark$$

$$y = \frac{ax^3}{3} + c$$

$x = 0, y = -1 \quad \therefore \quad -1 = 0 + c \quad \Rightarrow \quad c = -1$

$x = 1, y = 0 \quad \therefore \quad 0 = \frac{a}{3} - 1 \quad \checkmark$

$$3 = a$$

$$y = x^3 - 1 \quad \checkmark$$

See next page

Question 2 **(8 marks)**

If, at any time, a particle moves along a straight line with acceleration, a, cm/s^2, velocity, v, cm/s and displacement, x, cm from a fixed point O, the relationship between these variables is given by

$$a = \frac{dv}{dt} \text{ and } v = \frac{dx}{dt}$$

For the case when $a(t) = e^{2x}$ with initial conditions $x(0) = 1$ cm and $v(0) = e$ cm/s, determine

(a) the exact velocity of the particle when $x(t) = 5\ cm$. (4 marks)

$v \cdot \frac{dv}{dx} = e^{2x}$

$\int v\,dv = \int e^{2x}\,dx$

$\frac{v^2}{2} = \frac{e^{2x}}{2} + c$ ✓ integrates correctly

when $t=0, x=1, v=e$ $\frac{e^2}{2} = \frac{e^2}{2} + c \Rightarrow c = 0$ ✓ determines c

$\Rightarrow v^2 = e^{2x}$

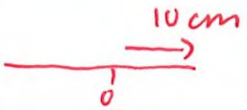
$v = \pm e^x$ but $x > 0, a > 0 \therefore v = e^x$ ✓ finds 'v'

when $x = 5$ $v = e^5$ cm/s ✓ calculates exact velocity

(b) the exact time when the particle will have a displacement of $10\ cm$ to the right of the origin. (4 marks)

$v = e^x$

$\frac{dx}{dt} = e^x$ ✓



sets up equation

$\int e^{-x}\,dx = \int dt$

$-e^{-x} = t + c$ ✓ ∫ correctly

when $t=0, x=1$ $-e^{-1} = 0 + c \Rightarrow c = -e^{-1}$

$\therefore -e^{-x} = t - e^{-1}$

$t = e^{-1} - e^{-x}$ ✓ determines 't' eqn

when $x = 10$ $t = e^{-1} - e^{-10}$ seconds ✓ calculates exact time.

End of questions

Additional working space

Question number: _____

Christ Church Grammar School

2019
TEST 5

MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	3 minutes
Working time for this section:	30 minutes
Marks available:	30 marks

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 3 **(7 marks)**

A particle moves along the $x-axis$ so that its displacement x, in metres, from the origin at any time t, in seconds, is given by:

$$x = 5 + 2\cos(4t)$$

(a) Show that the particle executes simple harmonic motion about some point P on the $x-axis$. (3 marks)

$x = 5 + 2\cos(4t)$

$\dot{x} = -8\sin(4t)$

$\ddot{x} = -32\cos(4t)$

$= -16(2\cos(4t))$

$= -4^2 x$

$\ddot{x} = -n^2 x$

$\therefore$ particle demonstrates SHM

(b) Find the point(s) on its path of motion where its speed is $4\ ms^{-1}$. (3 marks)

co-ordinate

$\frac{dx}{dt} = 4 = |-8\sin(4t)|$

$\Rightarrow \sin 4t = \pm\frac{1}{2}$

$4t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \ldots$ ✓

$t = \frac{\pi}{24}, \frac{5\pi}{24}, \ldots$

$x\left(\frac{\pi}{24}\right) = 5+\sqrt{3}$ ✓

$x\left(\frac{5\pi}{24}\right) = 5-\sqrt{3}$

$\therefore (5 \pm \sqrt{3}, 0)$

or $(6.732, 0)$ and $(3.268, 0)$

(c) Determine the acceleration of the particle when it is located $1\ m$ to the left of P. (1 mark)

$x = -1$

$\ddot{x} = -4^2 \times -1$

$= 16\ m/s^2$

See next page

Question 4 **(7 marks)**

The population $N(t)$ of numbats in a fauna reserve at time t years is modelled by the differential equation, $\frac{dN}{dt} = 0.000002N(500 - N)$ where $N(0) = 250$.

(a) Use integration to show that $N(t) = \frac{A}{1+Be^{-kt}}$, stating the values of A, B and k. (6 marks)

$$\int \frac{1}{N(500-N)}\,dN = 0.000002\int dt$$ ✓ sets up ∫

$$\Rightarrow \frac{1}{500}\int\left(\frac{1}{N} + \frac{1}{500-N}\right)dN = 0.000002\int dt$$ ✓ correct ∫ with partial fraction.

$$\frac{1}{N(500-N)} = \frac{A}{N} + \frac{B}{500-N}$$

$$\frac{1}{500}\left(\ln N + -\ln|500-N|\right) = 0.000002t + C$$ ✓ ∫ correctly

$1 = A(500-N) + BN$

$N=0 \quad 1 = 500A \quad \therefore A = \frac{1}{500}$

$N=500 \quad 1 = 500B \quad \therefore B = \frac{1}{500}$

$$\ln\left(\frac{N}{500-N}\right) = 0.001t + C$$

$$\frac{N}{500-N} = e^{0.001t+C}$$

$$\frac{N}{500-N} = Ae^{0.001t}$$ ✓ simplifies

When $t=0, N=250$ $\quad \frac{250}{250} = A = 1$

$$\frac{N}{500-N} = e^{0.001t}$$

$$N(1+e^{0.001t}) = 500e^{0.001t}$$

$$N = \frac{500e^{0.001t}}{1+e^{0.001t}}$$

$$N = \frac{500}{1+e^{-0.001t}}$$ ✓ simplifies expression

$\therefore A = 500, \; B = 1, \; k = +0.001$ ✓ explicit answer

See next page

Question 4 continued

(b) If current conditions prevail, what is the expected long-term population of numbats in the reserve? (1 mark)

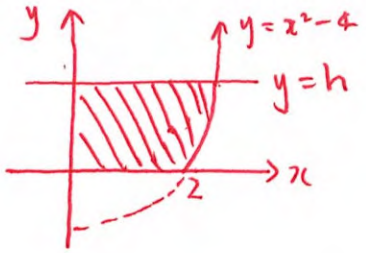
As $t \to \infty$ $N \to 500$

$\therefore$ long term population is 500 Numbats. ✓

Question 5 **(8 marks)**

The shape formed when the area in the first quadrant, which is enclosed between the $x-axis$, the $y-axis$, $y=x^2-4$ and $y=h$ $(h>0)$, is rotated about the $y-axis$, is the same as the interior of a flower vase.

(a) With one centimetre to one unit on each axis the value of h, to the nearest centimetre, if the capacity of the vase is $2.5\ L$. (4 marks)



$V = 2.5\,L$
$= 2500\,mL$
$= 2500\,cm^3$

If $y = x^2 - 4$
$x^2 = y + 4$

$\therefore V = \pi \int_0^h x^2\,dy$ ✓ sets up $\int$

$2500 = \pi \int_0^h (y+4)\,dy$

$\frac{2500}{\pi} = \frac{y^2}{2} + 4y \Big]_0^h$ ✓ correct $\int$ & = 2500

$\frac{2500}{\pi} = \frac{h^2}{2} + 4h - 0$

$\Rightarrow 0 = \frac{h^2}{2} + 4h - \frac{2500}{\pi}$

$\Rightarrow h = -44.0943$ or 36.0943 ✓ solve for 'h'

'h' > 0 $\therefore h = 36.0943$

$= 36\ cm$ (nearest cm) ✓ correct solution and rounding

Question 5 continued

(b) If water is poured into this vase at a constant rate of 5 cm^3/s find the rate at which the water level is rising when the depth of water in the vase is 6 cm.

(4 marks)

$\frac{dV}{dt} = 5$ $h = 6$ find $\frac{dh}{dt}$.

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ ✓ sets up chain rule

$= \frac{1}{\pi(h+4)} \times 5$ ✓ subst into formula

$V = \pi\left(\frac{h^2}{2} + 4h\right)$

$\frac{dV}{dh} = \pi(h+4)$ ✓ Finds $\frac{dh}{dV}$

$\frac{dh}{dV} = \frac{1}{\pi(h+4)}$

when $h = 6$

$\frac{dh}{dt} = \frac{5}{\pi(10)}$

$= \frac{1}{2\pi}$ cm/s. ✓ correct value when $h=6$.

(or 0.159155 cm/s) accept either answer with any rounding

See next page

Question 6 **(8 marks)**

An internet service provider plans to sample the volume of content downloaded per day by customers subscribing to their ADSL20 plan. From recent research, the company knew that the volume of downloads per customer was normally distributed with standard deviation 1.4 GB.

(a) Determine how large a sample the company should take in order to be 90% confident that the mean volume of downloads per customer calculated from their sample is within 0.25 GB of the true population mean. (2 marks)

$\sqrt{n} = \frac{1.645 \times 1.4}{0.25}$ ✓

$= 84.9$

$\therefore$ At least 85 as sample size ✓

answer only 2 marks

(b) A random selection of 25 subscribers was made and the total volume downloaded by these customers over a 24 hour period was 120GB. Calculate a 95% confidence interval for the mean volume of content downloaded per day by a customer. (4 marks)

$\bar{X} = \frac{120}{25}$

$= 4.8$ ✓

calculates $\bar{X}$ correctly

$4.8 \pm 1.96 \times \frac{1.4}{\sqrt{25}}$ ✓

$\therefore \quad 4.2512 < \mu < 5.3488$ ✓✓

one for each limit

accept any rounding

(c) If the company repeated the random sampling process and subsequent 95% confidence interval calculations from part (b) a total of 40 times, how many of the intervals calculated would you expect to contain the true population mean? Justify your answer. (2 marks)

95% of 40 = 38 ✓

$\therefore$ 95% of the confidence intervals will contain the population mean ✓

End of Questions

Additional working space

Question number: _____

Additional working space

Question number: _____